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Superintegrability on curved spaces, orbits and momentum hodographs: revisiting a classical result by Hamilton

José F Cariñena[1], **Manuel F Rañada**[1] **and Mariano Santander**[2]

[1] Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain
[2] Departamento de Física Teórica, Facultad de Ciencias, Universidad de Valladolid, 47011 Valladolid, Spain

E-mail: jfc@unizar.es, mfran@unizar.es and msn@fta.uva.es

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Abstract
The equation of the orbits (in the configuration space) and of the hodographs (in the 'momentum' plane) for the 'curved' Kepler and harmonic oscillator systems, living in a configuration space of any constant curvature and either signature type, are derived by purely algebraic means. This result extends to the 'curved' Kepler or harmonic oscillator for the classical Hamilton derivation of the orbits of the Euclidean Kepler problem through its hodographs. In both cases, the fundamental property allowing these derivations to work is the superintegrability of the 'curved' Kepler and harmonic oscillator, no matter whether the constant curvature of the configuration space is zero or not, or whether the configuration space metric is Riemannian or Lorentzian. In the 'curved' case the basic result does not refer to the 'velocity hodograph' but to the 'momentum hodograph'; both coincide in a Euclidean configuration space, but only the latter is unambiguously defined in all curved spaces.

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1. Introduction

The isotropic harmonic oscillator $V = \frac{1}{2}\omega_0^2 r^2$ and the Kepler potential $V = -k/r$ are two distinguished potentials in classical as well as in quantum mechanics. From the physical viewpoint, the harmonic oscillator arises as the osculating form of any potential around stable equilibria; it governs small oscillations and elementary quantum excitations. On the other hand, the Kepler potential dependence describes forces found in the solar system and in the hydrogen atom.

Both potentials are endowed with a mathematically important property: *superintegrability*. As mechanical systems on a 3D configuration space, they allow for the *maximal* number of functionally independent constants of motion (this number is 5 for a 6D phase space). The classical Bertrand theorem [1] states that the only central potentials all whose bounded (periodic) orbits are closed are precisely these two potentials. At the quantum level, these properties entail two facts: energy eigenvalues depend on a single quantum number and energy eigenfunctions can be explicitly computed (in several coordinate systems) in terms of classical functions of mathematical physics.

The existence of such a large set of conserved quantities (or operators) comes from hidden symmetries larger than the obvious $SO(3)$ one exhibited by the purely radial dependence. Rotational invariance leads to the constancy of angular momentum $\mathbf{J}$, thus the motion takes place in a 2D plane through the origin; it then suffices to consider the systems in 2D. Additionally, the Kepler potential has a specific vector constant of motion, the *Laplace vector*, usually known as the 'Runge–Lenz' vector [2–4], whose standard Cartesian components are

$$A_1 = J\dot{y} - k\cos\phi, \qquad A_2 = -J\dot{x} - k\sin\phi, \tag{1}$$

while the harmonic oscillator has a conserved *symmetric tensor* constant of motion, called the *Fradkin tensor* [5, 6], whose components are

$$F_{11} = (\dot{x})^2 + \omega_0^2 x^2, \qquad F_{12} = F_{21} = \dot{x}\dot{y} + \omega_0^2 xy, \qquad F_{22} = (\dot{y})^2 + \omega_0^2 y^2. \tag{2}$$

(the notation here is standard: (x, y) are Cartesian and (r, ϕ) are polar coordinates in the Euclidean plane, and J refers to the angular momentum, a scalar in 2D).

Related to their superintegrability, both the harmonic oscillator and Kepler potentials display separability for their Hamilton–Jacobi equations in *several* coordinate systems: this is called *multiseparability*. Both systems are trivially separable in polar coordinates; further to this the harmonic oscillator is separable in any member of a one-parameter family of Cartesian coordinates, with any orientation for the axes, and the Kepler problem is separable in a one-parameter family of parabolic coordinates, with a focus at the origin and any orientation for the axis [7]. While separability in polar coordinates is an obvious consequence of the most stringent rotational invariance, the additional separability exhibited by the harmonic oscillator and Kepler potentials is specific to the r^2 and $1/r$ radial dependences, and singles them out among the general radial potentials $V(r)$. Superintegrability and multiseparability are two rather fragile properties, which are destroyed by almost any perturbation in the functional form of the potential (we recall that there are however *non-central* superintegrable deformations of both potentials, as the Smorodinski–Winternitz potentials [8]).

We have recently shown [7, 9] that if the Euclidean configuration space is replaced by a space with *any constant curvature* and *any signature type* (a Cayley–Klein space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$), there exist potentials worthy of the name 'curved harmonic oscillator' and 'curved Kepler potential', which are *still superintegrable* in these spaces and reduce to the ordinary harmonic oscillator and the Kepler potential when the curvature of the configuration space vanishes and its signature is Riemannian. For the two Riemannian spaces (positive definite metric) with nonzero constant curvature κ—the sphere $\mathbf{S}^2_\kappa$ and the hyperbolic plane $\mathbf{H}^2_\kappa$—, the corresponding Kepler or harmonic oscillator potentials have been known since a long time; see historical references in [10].

Our approach uses a Cayley–Klein-type framework (hereafter CK), dealing simultaneously with all the spaces $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ in terms of two real parameters: κ_1 is the constant curvature and the metric is locally reducible to the form $\mathrm{diag}(1, \kappa_2)$. Then results can be obtained for all the configuration spaces $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ in a single run, all the expressions containing two free parameters κ_1, κ_2. Configuration spaces with a locally Minkowskian metric of any constant curvature are also included in this family. Motion in a Lorentzian configuration space has been much less explored [11], and even less in its anti-de Sitter or de Sitter curved relatives, so this 'CK parametric approach' opens views to a new field and goes beyond a unification of previously known results.

The outstanding properties of the Euclidean harmonic oscillator and Kepler potentials are indeed *generic* for the 'general curved' harmonic oscillator and the Kepler potentials *in any* $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$. In a recent note [12] we announced a complete, fully algebraic derivation of the *classical orbits* for the 'curved' Kepler potential in the CK space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ following *directly* from its superintegrable character. These orbits are always conics with a focus at the potential origin. This derivation is also closely related to a striking result by Hamilton [13]: for the Kepler motion in the Euclidean plane, hodographs are circles, and the conic character of the orbits can be easily derived (indeed this gives the simplest derivation, something which seems not to be as widely known as it should).

On the historical side, we must recall that (in the Euclidean case, of course) the idea to get the orbit starting from the specifically Keplerian-conserved quantities, thus bypassing any integration of motion equations, is however much older, antedating Hamilton by more than a century (see [2, 3]). We believe that the question on whether it is possible to extend this derivation to the Kepler motion in a curved *general CK configuration space* has not been discussed in the literature, perhaps because the *literal* extension, considering the *velocity hodographs*, does not make sense in curved spaces, where parallel transport of vectors is path dependent.

In this paper we obtain both *orbits* and *momentum hodographs* for the Kepler and harmonic oscillator in a general configuration space with any constant curvature and any signature type, and link this derivation to the superintegrability of these systems in a most direct way.

The derivation leans on two types of identities. One identity, among the Noether momenta in any $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$, is 'universal': it is independent of the potential and has the same explicit form in all the CK spaces $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$. The remaining identities are specific for the 'curved' Kepler and harmonic oscillator and appear as a consequence of the superintegrability of these systems. These identities are essentially the functional relations among the basic (quadratic) integrals of motion, which imply and are implied by some identities among functions defined in configuration space.

The scheme of the paper is as follows. In section 2 we provide all necessary details to deal with dynamics in a configuration CK space, by using the CK approach based on two parameters κ_1, κ_2. We introduce the Noether momenta, the 'universal' fundamental relation linking these momenta in $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$, and we discuss the (quadratic) superintegrability of these systems in terms of Noether momenta.

Section 3 is devoted to the Kepler problem in $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$, with the emphasis on the identities among some functions defined on the configuration space which are behind the superintegrability of the Kepler potential. Blending these identities with the fundamental identity among momenta in two different ways leads directly and algebraically to the equations of either configuration space orbits or momentum hodographs. Specialization to the case of a Euclidean configuration space recovers the already mentioned classical Hamilton approach, so section 3 can be considered as the 'curved' extension of Hamilton's results.

In section 4, the question is discussed for the harmonic oscillator; here the situation is slightly more complicated, as the additional conserved quantity is a symmetric tensor, but the logic in the derivation remains the same: superintegrability leads to some functions defined on a configuration space, satisfying some identities; blending these identities to the 'universal' fundamental one in two different ways gives both the configuration space orbits and the

Table 1. The nine standard two-dimensional CK spaces $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$.

	Measure of distance and sign of κ_1		
Measure of angle and sign of κ_2	Elliptic $\kappa_1 = 1$	Parabolic $\kappa_1 = 0$	Hyperbolic $\kappa_1 = -1$
Elliptic $\kappa_2 = 1$	Elliptic **S**^{2}	Euclidean **E**^{2}	Hyperbolic **H**^{2}
Parabolic $\kappa_2 = 0$	Co-Euclidean Oscillating NH **ANH**$^{1+1}$	Galilean **G**$^{1+1}$	Co-Minkowskian Expanding NH **NH**$^{1+1}$
Hyperbolic $\kappa_2 = -1$	Co-Hyperbolic Anti-de Sitter **AdS**$^{1+1}$	Minkowskian **M**$^{1+1}$	Doubly hyperbolic De Sitter **dS**$^{1+1}$

momentum hodographs for the 'curved' harmonic oscillator. Again, this can be considered as the 'curved' extension of the well-known properties of the harmonic oscillator in the Euclidean configuration space.

In particular, we show that the orbits for the 'curved' Kepler potential in the CK space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ are conics with a focus at the origin, while the momentum hodographs are *cycles*, the general 'curved' analogues of circles [14]. For the 'curved' harmonic oscillator in the general $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$, the orbits are centred conics in any $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ and the momentum hodographs are ellipses.

Section 5 gives a cursory look to the problem of extending the classical Bohlin transformation to the curved case. This extension exists and relates the curved Kepler problem to the curved harmonic oscillator in a way quite similar to the known $z \mapsto z^2$ Euclidean transformation [15, 16].

We close the paper with some comments in the last section.

To make the paper self-contained, two short appendices provide some basic information on non-Euclidean coordinates and conics in a general Cayley–Klein space.

2. Geometry and dynamics on a CK space

2.1. Geometry of a CK space

We denote by $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ a 2D space with constant curvature κ_1 and metric of a signature type $(1, \kappa_2)$. By rescaling lengths and angles, each of the two parameters κ_1, κ_2 can be brought to a *standard* value $1, 0, -1$ and the nine combinations correspond to the so-called standard Cayley–Klein CK spaces. These nine spaces can be conveniently displayed in table 1: the three rows accommodate spaces with either a Riemannian, degenerate and pseudo-Riemannian (Lorentzian) metric, according to the sign of κ_2, and along each row we find the spaces with constant curvature, positive, zero or negative. For more details see [17–19].

The symmetric homogeneous space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ admits a (maximal) three-dimensional isometry Lie group, denoted as $SO_{\kappa_1,\kappa_2}(3)$. This group is generated by a three-dimensional Lie algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ whose generators P_1, P_2 and J are given in the 'vector' matrix realization as

$$P_1 = \begin{pmatrix} 0 & -\kappa_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & -\kappa_1\kappa_2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa_2 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3}$$

with Lie commutation relations

$$[J, P_1] = P_2, \qquad [J, P_2] = -\kappa_2 P_1, \qquad [P_1, P_2] = \kappa_1 J. \tag{4}$$

When both constants are positive, the space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ is a two-dimensional sphere; the *standard* $\mathbf{S}^2$ corresponds to the choice $\kappa_1 = 1, \kappa_2 = 1$; other *standard* choices are $\kappa_1 = 0, \kappa_2 = 1 (\equiv \mathbf{E}^2)$ or $\kappa_1 = -1, \kappa_2 = 1 (\equiv \mathbf{H}^2)$, etc. The remaining six standard spaces are the anti-Newton–Hooke space $\mathbf{ANH}^{1+1}$, Galilean space $\mathbf{G}^{1+1}$, Newton–Hooke space $\mathbf{NH}^{1+1}$; anti-de Sitter sphere $\mathbf{AdS}^{1+1}$, Minkowskian space $\mathbf{M}^{1+1}$ and de Sitter sphere $\mathbf{dS}^{1+1}$.

In each case, (4) reduces to the corresponding commutation relations for the isometry algebras. When $\kappa_2 > 0$, the algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ is isomorphic to $\mathfrak{so}(3), \mathfrak{iso}(2), \mathfrak{so}(2,1)$ (related to the three Riemannian cases in the first row) according to the sign of κ_1. Likewise, when $\kappa_2 < 0$, the algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ is isomorphic to $\mathfrak{so}(2,1), \mathfrak{iso}(1,1), \mathfrak{so}(2,1)$ according to the sign of κ_1 (the three pseudo-Riemannian—Lorentzian—cases in the last row). The subgroup $SO_{\kappa_2}(2)$ generated by J has κ_2 as the unique label, and (P_1, P_2) behaves as a vector under these rotations; this subgroup $SO_{\kappa_2}(2)$ is isomorphic to either $SO(2), ISO(1), SO(1,1)$ according to the sign of κ_2.

We will use the κ-deformed 'cosine' $C_\kappa(x)$ and 'sine' $S_\kappa(x)$ functions:

$$C_\kappa(x) := \begin{cases} \cos\sqrt{\kappa}x \\ 1 \\ \cosh\sqrt{-\kappa}x, \end{cases} \qquad S_\kappa(x) := \begin{cases} \dfrac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}x & \kappa > 0 \\ x & \kappa = 0 \\ \dfrac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}x & \kappa < 0, \end{cases} \tag{5}$$

as well as the 'tangent' $T_\kappa(x) := S_\kappa(x)/C_\kappa(x)$. For $\kappa \neq 0$ these functions are (scaled) circular or hyperbolic trigonometric functions, reducing to them in the standard case $\kappa = \pm 1$. The intermediate functions which appear for $\kappa = 0$ are the 'parabolic cosine' equal to the constant function 1, and the 'parabolic sine' and 'parabolic tangent', both equal to identity linear function of their variables. Exponentials of matrices (3) lead to one-parametric subgroups $\exp(\alpha P_1), \exp(\beta P_2), \exp(\gamma J)$ of $SO_{\kappa_1,\kappa_2}(3)$ which can be expressed in terms of the labelled 'cosine' and 'sine' functions as

$$\begin{aligned} \exp(\alpha P_1) &= \begin{pmatrix} C_{\kappa_1}(\alpha) & -\kappa_1 S_{\kappa_1}(\alpha) & 0 \\ S_{\kappa_1}(\alpha) & C_{\kappa_1}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \exp(\beta P_2) &= \begin{pmatrix} C_{\kappa_1\kappa_2}(\beta) & 0 & -\kappa_1\kappa_2 S_{\kappa_1\kappa_2}(\beta) \\ 0 & 1 & 0 \\ S_{\kappa_1\kappa_2}(\beta) & 0 & C_{\kappa_1\kappa_2}(\beta) \end{pmatrix}, \\ \exp(\gamma J) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(\gamma) & -\kappa_2 S_{\kappa_2}(\gamma) \\ 0 & S_{\kappa_2}(\gamma) & C_{\kappa_2}(\gamma) \end{pmatrix}. \end{aligned} \tag{6}$$

This matrix group $SO_{\kappa_1,\kappa_2}(3)$ acts by matrix multiplication on an $\mathbb{R}^3$ *ambient space* by isometries of the 'ambient space metric':

$$\mathrm{d}s^2 = (\mathrm{d}s^0)^2 + \kappa_1(\mathrm{d}s^1)^2 + \kappa_1\kappa_2(\mathrm{d}s^2)^2, \tag{7}$$

and the space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ is the coset space $SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$, where $SO_{\kappa_2}(2)$ is the subgroup generated by J. The space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ can be described as the orbit of the point $(s^0, s^1, s^2) = (1, 0, 0)$ under the group action, an orbit Σ_0 which is contained in the 'sphere' $\Sigma \equiv (s^0)^2 + \kappa_1(s^1)^2 + \kappa_1\kappa_2(s^2)^2 = 1$. When $\kappa_1 \neq 0$, the natural metric on $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$, which will always be denoted as $\mathbf{g}$, is obtained from the metric induced on the orbit Σ_0 by the CK ambient space metric as

$$\mathbf{g} \equiv g_{\mu\nu}(q^1, q^2)dq^\mu dq^\nu = dl^2 = \frac{1}{\kappa_1}\mathrm{d}s^2|_\Sigma, \tag{8}$$

where $\mathrm{d}l^2$ is well defined even if $\kappa_1 \to 0$, because $\mathrm{d}s^2|_\Sigma$ also vanishes in this limit, with a well-defined quotient. With this metric, the scheme includes *simultaneously* the four well-known realizations of

- the standard sphere $\mathbf{S}^2$ (with a Riemannian metric of curvature 1) as the submanifold $(s^0)^2 + (s^1)^2 + (s^2)^2 = 1$ of the ambient 3D Euclidean space with $\mathrm{d}s^2 = (\mathrm{d}s^0)^2 + (\mathrm{d}s^1)^2 + (\mathrm{d}s^2)^2$;
- the standard hyperbolic plane $\mathbf{H}^2$ (with a Riemannian metric of curvature -1) as the submanifold $(s^0)^2 - (s^1)^2 - (s^2)^2 = 1$ of the ambient (1+2)D Minkowskian space $\mathrm{d}s^2 = (\mathrm{d}s^0)^2 - (\mathrm{d}s^1)^2 - (\mathrm{d}s^2)^2$, note that $\mathbf{g}$ is directly positive definite here;
- the anti-de Sitter sphere $\mathbf{AdS}^{1+1}$ (with a pseudo-Riemannian metric of curvature 1) as the submanifold $(s^0)^2 + (s^1)^2 - (s^2)^2 = 1$ of an ambient (2+1)d Minkowskian space with $\mathrm{d}s^2 = (\mathrm{d}s^0)^2 + (\mathrm{d}s^1)^2 - (\mathrm{d}s^2)^2$ and finally
- the de Sitter sphere $\mathbf{dS}^{1+1}$ (with a pseudo-Riemannian metric of curvature -1) as the submanifold $(s^0)^2 - (s^1)^2 + (s^2)^2 = 1$ of an ambient (2+1)D Minkowskian space $\mathrm{d}s^2 = (\mathrm{d}s^0)^2 - (\mathrm{d}s^1)^2 + (\mathrm{d}s^2)^2$.

Spaces with vanishing curvature (as $\mathbf{E}^2, \mathbf{M}^{1+1}$) are described here as *particular* cases: $\mathbf{E}^2$ corresponds to $\kappa_1 = 0, \kappa_2 = 1$, and $\mathbf{M}^{1+1}$ corresponds to $\kappa_1 = 0, \kappa_2 = -1$. In this CK approach all expressions and results implicitly depend upon the parameters κ_1, κ_2 in such a way that particularizing them to some values will always lead meaningfully to the expression or result in the corresponding geometry without any need for a limiting procedure or contraction.

The space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ can be referred to several systems of natural coordinates. The most important are the polar (r, ϕ), parallel '1' (x, v) and parallel '2' (u, y). These coordinates are described in appendix A, with figures A1 and A2 sketching the situation. A more detailed description of polar and parallel coordinates in the general $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ can be found in [20]. Here, it suffices to give the expressions of the ambient coordinates in terms of polar and parallel intrinsic coordinates:

$$\begin{pmatrix} s^0 \\ s^1 \\ s^2 \end{pmatrix} = \begin{pmatrix} C_{\kappa_1}(r) \\ S_{\kappa_1}(r)C_{\kappa_2}(\phi) \\ S_{\kappa_1}(r)S_{\kappa_2}(\phi) \end{pmatrix} = \begin{pmatrix} C_{\kappa_1}(x)C_{\kappa_1\kappa_2}(v) \\ S_{\kappa_1}(x) \\ C_{\kappa_1}(x)S_{\kappa_1\kappa_2}(v) \end{pmatrix} = \begin{pmatrix} C_{\kappa_1}(u)C_{\kappa_1\kappa_2}(y) \\ S_{\kappa_1}(u)C_{\kappa_1\kappa_2}(y) \\ S_{\kappa_1\kappa_2}(y) \end{pmatrix}, \tag{9}$$

and the metric, say $\mathrm{d}l^2 = g_{\mu\nu}(q^1, q^2)\,\mathrm{d}q^\mu\,\mathrm{d}q^\nu$ in general coordinates, is here given as

$$\mathrm{d}l^2 = \mathrm{d}r^2 + \kappa_2 S^2_{\kappa_1}(r)\,\mathrm{d}\phi^2 = \mathrm{d}x^2 + \kappa_2 C^2_{\kappa_1}(x)\,\mathrm{d}v^2 = C^2_{\kappa_1\kappa_2}(y)\,\mathrm{d}u^2 + \kappa_2\,\mathrm{d}y^2. \tag{10}$$

2.2. *The CK Noether momenta*

Consider now the motion of a particle in the configuration space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ under a natural mechanical-type Lagrangian, with a kinetic term given by the metric and possibly a potential depending on the coordinates:

$$\mathcal{L} = \tfrac{1}{2} g_{\mu\nu}(q^1, q^2) v_{q^\mu} v_{q^\nu} - \mathcal{V}(q^1, q^2). \tag{11}$$

Are there constants of motion for this Lagrangian which are *linear in the velocities?* This happens when the Lagrangian has a Killing vector field as an exact Noether symmetry. The Noether momenta are associated with the invariance under the three basic one-parameter subgroups in (6); these will also be denoted by P_1, P_2, J, and when computed as Lagrangian classical mechanics dictates [7] are given in terms of velocities as [20]

$$\begin{pmatrix} P_1 \\ P_2 \\ J \end{pmatrix} = \begin{pmatrix} C_{\kappa_2}(\phi)v_r - \kappa_2 C_{\kappa_1}(r)S_{\kappa_1}(r)S_{\kappa_2}(\phi)v_\phi \\ \kappa_2 S_{\kappa_2}(\phi)v_r + \kappa_2 C_{\kappa_1}(r)S_{\kappa_1}(r)C_{\kappa_2}(\phi)v_\phi \\ \kappa_2 S^2_{\kappa_1}(r)v_\phi \end{pmatrix}. \tag{12}$$

The momenta P_2 and J vanish *identically* when $\kappa_2 = 0$. This is obviously linked to the singular character of the corresponding Lagrangian, as the metric is degenerate when $\kappa_2 = 0$. But even in this singular case one may expect the geodesic motion to have precisely three non-trivial constants of motion linear in the velocities. This suggests us to consider the quantities defined as

$$\mathcal{P}_1 := P_1, \qquad \mathcal{P}_2 := \frac{P_2}{\kappa_2} \qquad \mathcal{J} := \frac{J}{\kappa_2}, \tag{13}$$

which when $\kappa_2 \neq 0$ are essentially equivalent to P_1, P_2, J but are instead well defined even if $\kappa_2 = 0$. In the following, we will refer to these new momenta as the *CK Noether momenta*:

$$\begin{pmatrix} \mathcal{P}_1 \\ \mathcal{P}_2 \\ \mathcal{J} \end{pmatrix} = \begin{pmatrix} C_{\kappa_2}(\phi)v_r - \kappa_2 C_{\kappa_1}(r)S_{\kappa_1}(r)S_{\kappa_2}(\phi)v_\phi \\ S_{\kappa_2}(\phi)v_r + C_{\kappa_1}(r)S_{\kappa_1}(r)C_{\kappa_2}(\phi)v_\phi \\ S^2_{\kappa_1}(r)v_\phi \end{pmatrix}. \tag{14}$$

The complete expressions in parallel coordinates are not required in this paper, but we shall need the expressions for $\mathcal{P}_1$ in 'parallel 1' coordinates (resp. $\mathcal{P}_2$ in 'parallel 2'):

$$\mathcal{P}_1 = C^2_{\kappa_1\kappa_2}(y)v_u, \qquad \mathcal{P}_2 = C^2_{\kappa_1}(x)v_v. \tag{15}$$

In the standard Euclidean $\mathbf{E}^2$, as a consequence of $\kappa_1 = 0$, the 'accidental' equalities among coordinates $u = x, v = y$ hold and the CK momenta $\mathcal{P}_1, \mathcal{P}_2$ reduce in Cartesian coordinates as they should to

$$\mathcal{P}_1|_{\mathbf{E}^2} = v_x = v_u, \qquad \mathcal{P}_2|_{\mathbf{E}^2} = v_v = v_y. \tag{16}$$

Back to the general CK case, the three CK momenta are linked, in all the CK spaces $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$, by a *fundamental* relation

$$s^2\mathcal{P}_1 - s^1\mathcal{P}_2 + s^0\mathcal{J} = 0, \tag{17}$$

which is 'universal' in a double sense: It does not depend on the potential, and it has the same form for all the CK spaces, i.e., it is explicitly independent of the parameters κ_1, κ_2. This relation will be a basic tool in our approach and can be checked directly. In the *standard* $\mathbf{E}^2$ case, this reduces to the well-known relation between angular and linear momentum $\mathcal{J} = x\mathcal{P}_2 - y\mathcal{P}_1 = xv_y - yv_x$. In any CK space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$, the relation (17) is well defined and allows us to express any of the three Noether momenta $\mathcal{P}_1, \mathcal{P}_2, \mathcal{J}$ in terms of the two remaining ones, with the coefficients depending on the coordinates.

2.3. Quadratic integrability and superintegrability in the CK spaces

Let us now focus our attention on potentials $\mathcal{V}(q^1, q^2)$ having constants of motion I *quadratic in the velocities*. The most general possible form for such a constant would be

$$I_\mathcal{K} = \mathcal{K}_{\mu\nu}(q^1, q^2)v_{q^\mu}v_{q^\nu} + \mathcal{W}(q^1, q^2). \tag{18}$$

The requirement for $I_\mathcal{K}$ to be a constant of motion translates into some conditions on $\mathcal{K}_{\mu\nu}$ and $\mathcal{W}$ to be satisfied for the given data $g_{\mu\nu}, \mathcal{V}$. These conditions naturally split into two subsets. The first subset is independent of the potential $\mathcal{V}$ and can be geometrically interpreted as stating that the tensor $\mathcal{K}_{\mu\nu}$ should be a *Killing tensor* for the metric $g_{\mu\nu}$. The most general solution to these equations in the generic case—and hence the most general Killing tensor in the generic CK space $\mathbf{S}^2_{\kappa_1[\kappa_2]}$—is

$$\mathcal{K}_{\mu\nu}(q^1, q^2) v_{q^\mu} v_{q^\nu} = a_0 \mathcal{J}^2 + a_1 \mathcal{P}_1^2 + a_2 \mathcal{P}_2^2 + 2a_{01} \mathcal{J} \mathcal{P}_1 + 2a_{02} \mathcal{J} \mathcal{P}_2 + 2a_{12} \mathcal{P}_1 \mathcal{P}_2, \tag{19}$$

where $a_0, a_1, a_2, a_{01}, a_{02}, a_{12}$ are arbitrary *constants*; this means that there is a 6D set of Killing tensors for all 2D CK spaces (the properties of Killing tensors defined in pseudo-Riemannian spaces are studied in [21, 22], with a geometrical classification for the Minkowski space). So for a constant of motion to be *quadratic in the velocities* is equivalent to be *quadratic in the CK momenta, with constant coefficients*. This extends to second order the well-known result: a constant of motion which is first order in the velocities necessarily equals to a linear combination of the three basic *CK momenta*, with *constant* coefficients.

Once a Killing tensor $\mathcal{K}_{\mu\nu}$ has been fixed, the second subset of conditions which ensure the constancy of (18) depends on the potential $\mathcal{V}$ and appears as a set of partial differential equations for $\mathcal{W}$ which may or may not admit a solution. The compatibility condition for this system is a single differential equation for the potential $\mathcal{V}$ containing as parameters the constants $a_0, a_1, a_2, a_{01}, a_{02}, a_{12}$ of $\mathcal{K}_{\mu\nu}$; its solutions are those potentials for which a constant of motion of the corresponding $I_{\mathcal{K}}$ type exists. If the potential $\mathcal{V}$ is equal to zero (or to any constant), then the compatibility conditions are *identically satisfied for any* $\mathcal{K}_{\mu\nu}$; this situation is to be expected as in this case there are three first integrals linear in the velocities, $\mathcal{J}, \mathcal{P}_1, \mathcal{P}_2$, whose six quadratic products are directly constants of I type (although 'non-primitive' ones) with $\mathcal{W} = 0$.

For an arbitrary potential $\mathcal{V}$ there is always a particular Killing tensor ($\mathcal{K} = \frac{1}{2} g$) which produces a constant of motion of type (18) with $\mathcal{W} = \mathcal{V}$: this is the (κ_1, κ_2)-'energy', denoted as I_E, whose part quadratic in the momenta is, up to a factor, the Casimir of the isometry group $SO_{\kappa_1\kappa_2}(3)$ [23]. The constant I_E expressed in terms of the Noether momenta is

$$I_E = \frac{1}{2} g_{\mu\nu}(q^1, q^2) v_{q^\mu} v_{q^\nu} + \mathcal{V}(q^1, q^2) = \frac{1}{2}\left(\mathcal{P}_1^2 + \kappa_2 \mathcal{P}_2^2 + \kappa_1\kappa_2 \mathcal{J}^2\right) + \mathcal{V}(q^1, q^2). \tag{20}$$

Only potentials with a particular structure allow *additional* quadratic constants of motion of I type. The characterization is the following [24, 25]: each Killing tensor determines a coordinate system of confocal type in the space $\mathbf{S}^2_{\kappa_1[\kappa_2]}$, and the potentials allowing a $I_{\mathcal{K}}$-type quadratic constant of motion are precisely those which are separable in this confocal coordinate system. For those $\mathcal{V}$ satisfying the compatibility condition, actual integration of the set of differential equations determining $\mathcal{W}$ can be dispensed with, as $\mathcal{W}$ can also be found 'a la Stäckel' through the separable expression for the potential in the associated confocal coordinates.

Superintegrable (quadratic) systems allow for (at least) two additional constants of this type, and hence the corresponding potentials are multiseparable; they are *very specific* potentials. The basics on the CK-type classification of superintegrable potentials has been advanced in [9]. As mentioned in the introduction, two superintegrable potentials which allow for the maximal number of additional constants are the 'harmonic oscillator' and the 'Kepler potential' in the CK space $\mathbf{S}^2_{\kappa_1[\kappa_2]}$. We discuss these in the next sections.

3. The Kepler potential in curved spaces

3.1. The 'curved' Kepler potential and its conserved quantities

In any $\mathbf{S}^2_{\kappa_1[\kappa_2]}$, the Kepler potential is defined to be [26, 27]

$$\mathcal{V}_{\mathrm{K}} = -\frac{k}{T_{\kappa_1}(r)}. \tag{21}$$

For the history of this potential, see references in [10, 28]. A recent book is devoted to the study of the Euclidean Kepler problem [29]. In the three Riemannian spaces of constant curvature (the CK spaces with $\kappa_2 > 0$), a rather physically natural criterion to provide a 'curved' version of the Kepler potential is to enforce the Gauss law in the corresponding CK three-dimensional space; this leads to the potential (21) which has therefore been known since a long time, starting from Lobachevsky himself. For quantum mechanics on the sphere, this potential resurfaced in the paper by Schrödinger [27], shortly to be followed by Infeld and Schild [26] who studied the hyperbolic space case.

This potential is superintegrable in any $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$. A first integral is linked to the invariance of $\mathcal{V}_{\rm K}$ under rotations around the potential centre and leads to the constancy of angular momentum $\mathcal{J}$, and to the quadratic constant $I_{\mathcal{J}^2} = \mathcal{J}^2$. In any $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$, the potential (21) allows for *two* additional constants of motion of types $I_{\mathcal{J}\mathcal{P}_1}$, $I_{\mathcal{J}\mathcal{P}_2}$, which are associated with the separability of the Kepler potential in two equiparabolic '01' and '20' coordinate systems (see [7]). These constants of motion are

$$\begin{aligned} I_{\mathcal{J}\mathcal{P}_1} &= \mathcal{J}\mathcal{P}_1 + \mathcal{W}_{01}, \qquad & \mathcal{W}_{01} &= k S_{\kappa_2}(\phi) \\ I_{\mathcal{J}\mathcal{P}_2} &= \mathcal{J}\mathcal{P}_2 + \mathcal{W}_{02}, \qquad & \mathcal{W}_{02} &= k V_{\kappa_2}(\phi), \end{aligned} \tag{22}$$

where $V_{\kappa_2}(\phi)$ is the CK version of the 'versed sine', $V_{\kappa_2}(\phi) = (1 - C_{\kappa_2}(\phi))/\kappa_2$; for $\kappa_2 = 0$ this reduces to the function $\phi^2/2$. Checking that $I_{\mathcal{J}\mathcal{P}_1}$, $I_{\mathcal{J}\mathcal{P}_2}$ are constants of motion can be carried out by a simple direct computation. A point which might easily pass unnoticed is the existence of an identity, a consequence of $C^2_{\kappa_2}(\phi) + \kappa_2 S^2_{\kappa_2}(\phi) = 1$, among the functions $\mathcal{W}_{01}$, $\mathcal{W}_{02}$:

$$(\mathcal{W}_{01})^2 + \mathcal{W}_{02}\,(\kappa_2 \mathcal{W}_{02} - 2) = 0. \tag{23}$$

The two constants $I_{\mathcal{J}\mathcal{P}_1}$, $I_{\mathcal{J}\mathcal{P}_2}$, together with the energy and the angular momentum,

$$I_E = \frac{1}{2}\big(\mathcal{P}_1^2 + \kappa_2 \mathcal{P}_2^2 + \kappa_1\kappa_2 \mathcal{J}^2\big) - \frac{k}{T_{\kappa_1}(r)}, \qquad I_{\mathcal{J}^2} = \mathcal{J}^2, \tag{24}$$

provide a set of four constants of motion. As the maximal number of functionally independent constants of motion for this system is 3, a single relation among the four I above should exist. This relation which is quadratic in I's can be checked using (23) and reads

$$(I_{\mathcal{J}\mathcal{P}_1})^2 + I_{\mathcal{J}\mathcal{P}_2}(\kappa_2 I_{\mathcal{J}\mathcal{P}_2} - 2k) = \big(2I_E - \kappa_1\kappa_2 I_{\mathcal{J}^2}\big)\, I_{\mathcal{J}^2}. \tag{25}$$

The two constants $I_{\mathcal{J}\mathcal{P}_1}$, $I_{\mathcal{J}\mathcal{P}_2}$ can be seen as the components of a single Keplerian (conserved) vector under the (sub)group $SO_{\kappa_2}(2)$ of rotations around the origin in the space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$. This vector will be called here the *CK eccentricity vector* $\boldsymbol{\mathcal{E}}$; along any evolution under the Kepler potential, the (constant) values of the components of $\boldsymbol{\mathcal{E}}$ will be denoted by $\mathcal{E}_{01}$, $\mathcal{E}_{02}$ and reads

$$\begin{pmatrix} \mathcal{E}_{01} \\ \mathcal{E}_{02} \end{pmatrix} = \begin{pmatrix} I_{\mathcal{J}\mathcal{P}_1} \\ I_{\mathcal{J}\mathcal{P}_2} \end{pmatrix} = \begin{pmatrix} \mathcal{J}\mathcal{P}_1 + \mathcal{W}_{01} \\ \mathcal{J}\mathcal{P}_2 + \mathcal{W}_{02} \end{pmatrix}, \tag{26}$$

and in terms of $\mathcal{E}_{01}$, $\mathcal{E}_{02}$ and the values of energy and angular momentum, (25) is written as

$$\mathcal{E}_{01}^2 + \mathcal{E}_{02}(\kappa_2 \mathcal{E}_{02} - 2k) = (2E - \kappa_1\kappa_2 \mathcal{J}^2)\mathcal{J}^2, \tag{27}$$

which is well defined in all the CK spaces $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$. We will later describe the relation of $\boldsymbol{\mathcal{E}}$ to the ordinary Laplace–Runge–Lenz vector in the Euclidean case. For the moment, note the appearance on the rhs of the specific combination $(2E - \kappa_1\kappa_2\mathcal{J}^2)$, which for the Euclidean space reduces to $2E$.

To sum up, the existence of a Keplerian additional conserved vector is not a specifically Euclidean property, but still holds even if the configuration space is the general $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$, with any constant curvature and *any signature*. The 'Riemannian' part of this statement has been known since a long time for the Kepler problem in $\mathbf{S}^2_{\kappa_1}$ and $\mathbf{H}^2_{\kappa_1}$; these cases appear in our approach when $\kappa_2 = 1$. The CK formalism also covers the cases where $\kappa_2 < 0$, this is the Kepler problem in a locally Minkowskian constant curvature configuration space, a case which definitely has not been studied before. The existence of this vector constant is not specific for the 2D case. For the motion of a particle under the 'curved' Kepler potential in a 'curved' *N*D configuration space of Cayley–Klein type, there exists a 'curved' form for the Laplace–Runge–Lenz vector, which for any N has been obtained by Herranz and Ballesteros [30]. Thus, the existence of a (3D or even *N*D Euclidean) Laplace–Runge–Lenz vector is not a specifically Euclidean property (note however that [30] uses the momenta P_1, P_2 and thus extra factors κ_2 appear at places when compared with the expressions for $\mathcal{P}_1, \mathcal{P}_2$ used in the present paper).

3.2. The orbits of a particle moving in a 'curved' Kepler potential

The superintegrability of the Kepler problem on any $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ has a direct bearing on the orbits. For the Kepler problem in the Euclidean plane, both Laplace (1799) and Hamilton (1845), preceded by J Hermann (or Ermanno, 1710) and J I Bernoulli (1710) [2, 3, 31], were able to derive the nature of the orbits as conics with a focus at the potential centre *directly* from the existence of two new constants of motion. This derivation of the Kepler orbits does not seem to be widely known. The superintegrability of the problem is also connected to another beautiful result by Hamilton [13]: the 'hodographs' of the Euclidean Kepler problem are circles in the velocity space (see the papers by Milnor [32] and Anosov [33] and references therein).

With some suitable shift in the interpretation, both results can be extended from the Euclidean Kepler problem to its general 'curved' version. Let us start by finding the 'curved' Kepler orbits in any $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$. As here $\mathcal{J}$ itself is also a first integral, we may express both $\mathcal{P}_1$ and $\mathcal{P}_2$ in terms of the values $\mathcal{E}_{01}, \mathcal{E}_{02}, \mathcal{J}$ (which will be constants along any motion) and $\mathcal{W}$ (which depends only on the coordinates):

$$\mathcal{P}_1 = \frac{1}{\mathcal{J}}(\mathcal{E}_{01} - \mathcal{W}_{01}) \qquad \mathcal{P}_2 = \frac{1}{\mathcal{J}}(\mathcal{E}_{02} - \mathcal{W}_{02}). \tag{28}$$

Enforcing the universal fundamental relation (17) in the form $s^0\mathcal{J} = s^1\mathcal{P}_2 - s^2\mathcal{P}_1$ gives

$$s^0\mathcal{J}^2 = s^1(\mathcal{E}_{02} - \mathcal{W}_{02}) - s^2(\mathcal{E}_{01} - \mathcal{W}_{01}), \tag{29}$$

but using (23) the terms coming from the functions $\mathcal{W}_{01}, \mathcal{W}_{02}$ simplify

$$\begin{aligned} -s^1\mathcal{W}_{02} + s^2\mathcal{W}_{01} &= -s^1 k V_{\kappa_2}(\phi) + s^2 k S_{\kappa_2}(\phi) = k S_{\kappa_1}(r)\left(-C_{\kappa_2}(\phi)V_{\kappa_2}(\phi) + S^2_{\kappa_2}(\phi)\right) \\ &= k S_{\kappa_1}(r)\left(\frac{1 - C_{\kappa_2}(\phi)}{\kappa_2}\right) = k S_{\kappa_1}(r) V_{\kappa_2}(\phi) \\ &= k\left(\frac{\sqrt{(s^1)^2 + \kappa_2(s^2)^2} - s^1}{\kappa_2}\right). \end{aligned} \tag{30}$$

The result is well defined even when $\kappa_2 = 0$ as it is seen clearly in some of the alternative forms in the previous expression. Thus, we finally get the orbit equation in the ambient space coordinates:

$$s^0\mathcal{J}^2 = s^1\mathcal{E}_{02} + s^2\mathcal{E}_{01} + k\left(\frac{\sqrt{(s^1)^2 + \kappa_2(s^2)^2} - s^1}{\kappa_2}\right). \tag{31}$$

This equation can be translated to any other coordinate system by using (9). In particular, in polar coordinates, this equation is

$$\frac{1}{T_{\kappa_1}(r)} = \frac{1}{\mathcal{J}^2}\left(\mathcal{E}_{02}C_{\kappa_2}(\phi) + \mathcal{E}_{01}S_{\kappa_2}(\phi) + kV_{\kappa_2}(\phi)\right), \tag{32}$$

and in all $\mathbf{S}^2_{\kappa_1[\kappa_2]}$ corresponds to a CK conic (i.e., a conic in the intrinsic sense of the geometry of the CK space) with a focus at the origin. Under the previous form, the equation applies for *all* CK spaces. A slight simplification is possible when $\kappa_2 \neq 0$, because in this case the sum of a 'cosine' and 'versed sine' can be rewritten as the sum of a 'constant' and a 'cosine', at the price of an (only apparent) indetermination when $\kappa_2 = 0$:

$$\frac{1}{T_{\kappa_1}(r)} = \frac{1}{\kappa_2 \mathcal{J}^2}\big(k + (\kappa_2 \mathcal{E}_{02} - k)\, C_{\kappa_2}(\phi) + \kappa_2 \mathcal{E}_{01} S_{\kappa_2}(\phi)\big). \tag{33}$$

The coefficients of $C_{\kappa_2}(\phi)$ and $S_{\kappa_2}(\phi)$ in this equation are the components of a vector to be denoted as $\boldsymbol{\mathcal{A}}$, the *CK Laplace–Runge–Lenz vector*:

$$\begin{pmatrix}\mathcal{A}_1\\ \mathcal{A}_2\end{pmatrix} := \begin{pmatrix}\kappa_2 \mathcal{E}_{02} - k\\ -\mathcal{E}_{01}\end{pmatrix} = \begin{pmatrix}\kappa_2\, (\mathcal{J}\mathcal{P}_2 + \mathcal{W}_{02}) - k\\ -\mathcal{J}\mathcal{P}_1 - \mathcal{W}_{01}\end{pmatrix}. \tag{34}$$

In the Euclidean plane this vector reduces precisely to the Laplace–Runge–Lenz vector. In intrinsic terms, the CK Laplace–Runge–Lenz vector is the Hodge κ_2-dual of the eccentricity vector (κ_2 in the name recalls that the Hodge dual is taken in a flat plane with metric of signature κ_2) shifted by the constant vector $(-k, 0)$:

$$\begin{pmatrix}\mathcal{E}_{01}\\ \mathcal{E}_{02}\end{pmatrix} \to \begin{pmatrix}\kappa_2\mathcal{E}_{02}\\ -\mathcal{E}_{01}\end{pmatrix} \to \begin{pmatrix}\kappa_2\mathcal{E}_{02}\\ -\mathcal{E}_{01}\end{pmatrix} - \begin{pmatrix}k\\ 0\end{pmatrix} = \begin{pmatrix}\mathcal{A}_1\\ \mathcal{A}_2\end{pmatrix}. \tag{35}$$

When compared with the eccentricity vector, a point is worth noting: the eccentricity vector (26) contains the relevant information on orbits for all CK spaces, but through the (κ_2)-Hodge-type duality and shifting involved in the transition to the vector $\boldsymbol{\mathcal{A}}$, the Laplace–Runge–Lenz loses a part of this information when $\kappa_2 = 0$; in this case the component $\mathcal{A}_1$ has a value $-k$ independently of the motion. Hence, the eccentricity vector should be looked at as the 'preferred CK form' of the conserved Keplerian vector.

The naming here keeps the established name *'Laplace–Runge–Lenz'* for the general CK version (34) whose Euclidean specialization is the standard LRL vector and reserves the name *'eccentricity' vector* for the general CK version of the Hamilton vector, which is meaningful for all CK spaces. In the Euclidean case, [31] contains a lot of information on these vectors and their variants in different systems.

The general form of the Kepler orbits in $\mathbf{S}^2_{\kappa_1[\kappa_2]}$ is (32, 33), which represents a conic in the CK space, with a focus at the origin. The type of the conic depends on the CK space and on the values of the physical constants $\mathcal{E}_{01}, \mathcal{E}_{02}$ (or $\mathcal{A}_1, \mathcal{A}_2$). The family (33) includes, in any CK space with $\kappa_2 \neq 0$, conics with CK eccentricity e, semilatus rectum p and orientation at the periastron ϕ_0:

$$\begin{aligned}\frac{1}{T_{\kappa_1}(r)} &= \frac{1}{\sqrt{\kappa_2}\, T_{\kappa_1\kappa_2}(p)}\,\big(1 + eC_{\kappa_2}(\phi - \phi_0)\big)\\ &= \frac{1}{\sqrt{\kappa_2}\, T_{\kappa_1\kappa_2}(p)}\big(1 + eC_{\kappa_2}(\phi)C_{\kappa_2}(\phi_0) + e\kappa_2 S_{\kappa_2}(\phi)S_{\kappa_2}(\phi_0)\big)\end{aligned} \tag{36}$$

and the relation between geometric and physical constants is

$$(\mathcal{A}_1, \mathcal{A}_2) = ke(C_{\kappa_2}(\phi_0), -S_{\kappa_2}(\phi_0)), \qquad \sqrt{\kappa_2}\, T_{\kappa_1\kappa_2}(p) = \frac{\kappa_2 \mathcal{J}^2}{k}, \qquad \mathcal{A}_1^2 + \kappa_2 \mathcal{A}_2^2 = k^2 e^2. \tag{37}$$

The last relation can be compared with (27) for $\boldsymbol{\mathcal{E}}$, translated into terms of $\boldsymbol{\mathcal{A}}$:

$$\mathcal{A}_1^2 + \kappa_2 \mathcal{A}_2^2 = k^2 + (2E - \kappa_1\kappa_2 \mathcal{J}^2)\kappa_2 \mathcal{J}^2, \tag{38}$$

leading to the general relation among e and the energy and angular momentum:

$$e^2 = 1 + \frac{(2E - \kappa_1\kappa_2\mathcal{J}^2)\kappa_2\mathcal{J}^2}{k^2}. \tag{39}$$

All these previous relations reduce in the standard Euclidean plane (with the notational changes $(\mathcal{A}_1, \mathcal{A}_2)|_{\mathbf{E}^2} \equiv (A_1, A_2)$ and $\mathcal{J}|_{\mathbf{E}^2} \equiv J$) to the well known ones

$$\left(A_1^2 + A_2^2\right) = k^2 + 2EJ^2, \qquad \|\mathbf{A}\| = ke, \qquad e = \left(1 + \frac{2EJ^2}{k^2}\right)^{1/2}. \tag{40}$$

In any particular CK space, this family may include several types of conics. For instance, in the Euclidean plane $\mathbf{E}^2$ ellipses appear for $0 < e < 1$, but the family (36) also includes parabolas for $e = 1$ and hyperbolas for $e > 1$; in $\mathbf{E}^2$ the class of the conic depends only on e, but is independent of p. In the hyperbolic plane $\mathbf{H}^2$ the family (36) includes ellipses, horoellipses, parabolas, horohyperbolas and hyperbolas, and in this case the class of the conic given in (36) depends both on e and p, see [34].

If the polar axis is oriented along the 'periastron', then both the LRL vector and the eccentricity vectors are carried to some standard reference positions:

$$\mathcal{A}_1 = ke, \qquad \mathcal{A}_2 = 0, \qquad \mathcal{E}_{01} = 0, \qquad \mathcal{E}_{02} = \frac{k(e+1)}{\kappa_2}. \tag{41}$$

All these equations are in full agreement with results previously known for the Kepler problem in the three standard ($\kappa_2 = 1$) Riemannian configuration spaces of constant curvature (see [34] and references therein). For the 'Kepler motion' in constant curvature locally Minkowskian spaces ($\kappa_2 < 0$) these results seem to be new.

3.3. Momentum Hodographs for the 'curved' Kepler potentials are 'cycles'

In this subsection, we look for a natural extension of the celebrated Hamilton result for the Kepler problem. Essentially what Hamilton found was the following property: when the particle moves in the Euclidean plane according to Newton's equations under the Kepler potential, its velocity vector $(\dot{x}, \dot{y})$ traces out a (Euclidean) circle in the *velocity space*. Using this property it is possible to derive the orbits in a surprisingly direct way. The connection with superintegrability is also direct, though not always mentioned: the circle character of hodographs is equivalent to the constancy (in the Euclidean case) of the two quantities (1) (see [32]).

The naive attempt to extend this result for the 'velocity vector' in the Kepler problem on a *curved* configuration space soon finds a blocking stone: there is not any canonical way to translate this velocity vector at each point of the trajectory to a common 'origin' in the velocity space. A seemingly natural possibility, the canonical parallel transport in a configuration space, produces different results if done along different paths. Thus, thinking in terms of the velocity vector does not seem to be a good standpoint to search for an extension of Hamilton's 'hodograph' result to a space with nonzero constant curvature.

But in the Euclidean space the components of the velocity *coincide* with the values of the two Noether momenta $\mathcal{P}_1, \mathcal{P}_2$, as follows from the relation (16) in the standard Euclidean case $\mathbf{E}^2$. Hence, (34) can be rewritten in the Euclidean standard case as

$$\begin{aligned} \mathcal{A}_1|_{\mathbf{E}^2} &= \kappa_2(\mathcal{J}\mathcal{P}_2 + kV_{\kappa_2}(\phi))|_{\mathbf{E}^2} - k = J\dot{y} + k(1-\cos\phi) - k = J\dot{y} - k\cos\phi \\ \mathcal{A}_2|_{\mathbf{E}^2} &= (-\mathcal{J}\mathcal{P}_1 - kS_{\kappa_2}(\phi))|_{\mathbf{E}^2} = -J\dot{x} - k\sin\phi, \end{aligned} \tag{42}$$

which is precisely the well-known form given in the introduction (1). We recall that it suffices to enforce the Euclidean relation $J = x\dot{y} - y\dot{x}$ with $\dot{x}, \dot{y}$ taken from (1) to get the orbit equation;

this was the path followed by Hamilton, who arrived at these constants as a consequence of the circle character of the hodograph, which he established directly from the Newton's equations and the law of areas.

Back to the general CK case, the CK momenta $\mathcal{P}_1, \mathcal{P}_2$ are defined for any curvature and signature type of the configuration space, and if in (28) we replace the functions $\mathcal{W}$ by their expressions, we obtain the parametric equations of the hodograph, with the polar angle as a parameter:

$$\mathcal{P}_1 = \frac{1}{\mathcal{J}}\big(\mathcal{E}_{01} - k S_{\kappa_2}(\phi)\big) \qquad \mathcal{P}_2 = \frac{1}{\mathcal{J}}\big(\mathcal{E}_{02} - k V_{\kappa_2}(\phi)\big), \tag{43}$$

The hodograph equation can be obtained most directly by enforcing the identity (23) for $\mathcal{W}_{01} = \mathcal{E}_{01} - \mathcal{P}_1\mathcal{J}$ and $\mathcal{W}_{02} = \mathcal{E}_{02} - \mathcal{P}_2\mathcal{J}$; after expanding and simplifying, this gives

$$\left(\mathcal{P}_1 - \frac{\mathcal{E}_{01}}{\mathcal{J}}\right)^2 + \kappa_2\left(\mathcal{P}_2 - \frac{\mathcal{E}_{02} - \frac{k}{\kappa_2}}{\mathcal{J}}\right)^2 = \frac{k^2}{\kappa_2\mathcal{J}^2}, \tag{44}$$

which is the equation of a cycle in the $(\mathcal{P}_1, \mathcal{P}_2)$-plane, with centre $(\mathcal{E}_{01}/\mathcal{J}, (\mathcal{E}_{02} - k/\kappa_2)\mathcal{J})$ and 'radius' $k/(\sqrt{\kappa_2}\mathcal{J})$, relative to the natural geometry of the $\mathcal{P}$-plane which is based on the quadratic form $\mathcal{P}_1^2 + \kappa_2\mathcal{P}_2^2$, as suggested by (20); this intrinsic geometry is *independent* of the curvature κ_1 and depends *only* on the signature label κ_2. In this $(\mathcal{P}_1, \mathcal{P}_2)$-plane (whose standard forms are $\mathbf{E}^2$, $\mathbf{G}^{1+1}$, $\mathbf{M}^{1+1}$ according to the three standard choices for κ_2), cycles mean 'curves of constant curvature'; in $\mathbf{E}^2$ these curves are circles, with straight lines as a limit case.

Hence if the configuration space is a classical Riemannian space ($\kappa_2 = 1$) of constant curvature, and the term 'hodograph' is understood as the curve traced out in the (Euclidean) $(\mathcal{P}_1, \mathcal{P}_2)$-plane, then this curve is a standard Euclidean cycle (either a circle or a straight line) regardless of the value of the curvature κ_1 of the configuration space. And if the configuration space is a Lorentzian space ($\kappa_2 = -1$) of constant curvature (including the Minkowski space as $\kappa_1 = 0$), then the 'hodograph' traces out (some arc of) a 'cycle' in the Minkowskian $(\mathcal{P}_1, \mathcal{P}_2)$-plane; these Minkowskian cycles appear to the affine eye as equilateral hyperbolas. Finally, when $\kappa_2 = 0$ then for any κ_1 the $(\mathcal{P}_1, \mathcal{P}_2)$-plane has a Galilean geometry, and the cycles are either special lines or affine parabolas with special axis.

4. The harmonic oscillator in curved spaces

In any CK space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ of constant curvature the 'harmonic oscillator' (HO) potential is defined to be:

$$\mathcal{V}_{\mathrm{HO}} = \tfrac{1}{2}\omega_0^2 T^2_{\kappa_1}(r). \tag{45}$$

In the sphere $\mathbf{S}^2$ this potential was first considered at the end of 19th century [35] and was studied later on by Higgs [36] and Leemon [37]. Let us first give the expressions of the potential in the three basic coordinate systems (see [7]):

$$T^2_{\kappa_1}(r) = T^2_{\kappa_1}(x) + \kappa_2\frac{T^2_{\kappa_1\kappa_2}(v)}{C^2_{\kappa_1}(x)} = \frac{T^2_{\kappa_1}(u)}{C^2_{\kappa_1\kappa_2}(y)} + \kappa_2 T^2_{\kappa_1\kappa_2}(y). \tag{46}$$

These relations display the separability of the HO not only in polar, but also in parallel '1' and '2' coordinates ($\mathcal{V}_{\mathrm{HO}}$ also allows separability in equiparabolic '12' coordinates). This means that, additionally to the energy, the motion of a 'curved' harmonic oscillator has constants of motion of types $I_{\mathcal{J}^2}, I_{\mathcal{P}_1^2}, I_{\mathcal{P}_2^2}, I_{\mathcal{P}_1\mathcal{P}_2}$. The full expressions are

$$\begin{aligned} &I_{\mathcal{J}^2} = \mathcal{J}^2 && I_{\mathcal{P}_1^2} = \mathcal{P}_1^2 + \mathcal{W}_{11}(q_1, q_2) \\ &I_{\mathcal{P}_2^2} = \mathcal{P}_2^2 + \mathcal{W}_{22}(q_1, q_2) && I_{\mathcal{P}_1\mathcal{P}_2} = \mathcal{P}_1\mathcal{P}_2 + \mathcal{W}_{12}(q_1, q_2), \end{aligned} \tag{47}$$

with $\mathcal{W}_{11}(q_1, q_2)$, $\mathcal{W}_{22}(q_1, q_2)$, $\mathcal{W}_{12}(q_1, q_2)$ given in the three coordinate systems by

$$\begin{aligned}
\mathcal{W}_{11} &= \omega_0^2 T_{\kappa_1}^2(r) C_{\kappa_2}^2(\phi) = \omega_0^2 \frac{T_{\kappa_1}^2(x)}{C_{\kappa_1\kappa_2}^2(v)} = \omega_0^2 T_{\kappa_1}^2(u) \\
\mathcal{W}_{22} &= \omega_0^2 T_{\kappa_1}^2(r) S_{\kappa_2}^2(\phi) = \omega_0^2 T_{\kappa_1\kappa_2}^2(v) = \omega_0^2 \frac{T_{\kappa_1\kappa_2}^2(y)}{C_{\kappa_1}^2(u)} \\
\mathcal{W}_{12} &= \omega_0^2 T_{\kappa_1}^2(r) C_{\kappa_2}(\phi) S_{\kappa_2}(\phi) = \omega_0^2 \frac{T_{\kappa_1}(x) T_{\kappa_1\kappa_2}(v)}{C_{\kappa_1\kappa_2}(v)} = \omega_0^2 \frac{T_{\kappa_1}(u) T_{\kappa_1\kappa_2}(y)}{C_{\kappa_1}(u)} = \omega_0^2 T_{\kappa_1}(u) T_{\kappa_1\kappa_2}(v).
\end{aligned} \tag{48}$$

Remark that $\mathcal{W}_{11}, \mathcal{W}_{22}, \mathcal{W}_{12}$ are well defined for any CK space and they do not vanish identically in none of them. In terms of the ambient space coordinates, these functions are

$$\begin{aligned}
\mathcal{V}_{\mathrm{HO}} &= \omega_0^2 \frac{(s^1)^2 + \kappa_2 (s^2)^2}{(s^0)^2}, \\
\mathcal{W}_{11} &= \omega_0^2 \frac{(s^1)^2}{(s^0)^2}, \qquad \mathcal{W}_{22} = \omega_0^2 \frac{(s^2)^2}{(s^0)^2}, \qquad \mathcal{W}_{12} = \omega_0^2 \frac{(s^1)(s^2)}{(s^0)^2}.
\end{aligned} \tag{49}$$

We recall here the existence of two identities among the functions $\mathcal{W}_{ij}$ and the potential:

$$\begin{aligned}
&\tfrac{1}{2}(\mathcal{W}_{11} + \kappa_2 \mathcal{W}_{22}) = \mathcal{V}_{\mathrm{HO}} \\
&\mathcal{W}_{11}\mathcal{W}_{22} - \mathcal{W}_{12}^2 = 0.
\end{aligned} \tag{50}$$

These equations can be checked directly. Both are obvious in terms of the ambient space coordinates (49); the first also follows as a consequence of the basic identity between $C_{\kappa_2}(\phi)$ and $S_{\kappa_2}(\phi)$ or, alternatively, as a consequence of the following *identity* holding for all values of its arguments (a, b), applied to the two pairs of variables (x, v) and (u, y):

$$T_{\kappa_1}^2(a) + \kappa_2 \frac{T_{\kappa_1\kappa_2}^2(b)}{C_{\kappa_1}^2(a)} = \frac{T_{\kappa_1}^2(a)}{C_{\kappa_1\kappa_2}^2(b)} + \kappa_2 T_{\kappa_1\kappa_2}^2(b) \tag{51}$$

Together with the energy I_E there are thus *five* different constants of I type for the HO motion. The maximum number of functionally independent constants in a system with a 2D configuration space is 3, thus one may expect *two* functionally independent relations among the five I constants. These can be established using the two identities (50):

$$\begin{aligned}
&I_E = \tfrac{1}{2}\left(I_{\mathcal{P}_1^2} + \kappa_2 I_{\mathcal{P}_2^2} + \kappa_1\kappa_2 I_{\mathcal{J}^2}\right) \\
&I_{\mathcal{P}_1^2} I_{\mathcal{P}_2^2} - \left(I_{\mathcal{P}_1\mathcal{P}_2}\right)^2 = \omega_0^2 I_{\mathcal{J}^2}.
\end{aligned} \tag{52}$$

The first equation in (52) is the linear relation between the energy constant and three constants $I_{\mathcal{P}_1^2}, I_{\mathcal{P}_2^2}, I_{\mathcal{J}^2}$ (when any system allows three separate constants of motion of the types $I_{\mathcal{P}_1^2}, I_{\mathcal{P}_2^2}, I_{\mathcal{J}^2}$, this relation with the energy should necessarily hold). The second relation in (52) is a *quadratic* relation among $I_{\mathcal{P}_1^2}, I_{\mathcal{P}_2^2}, I_{\mathcal{P}_1\mathcal{P}_2}, I_{\mathcal{J}^2}$, which is well known in the Euclidean oscillator case; surprisingly this relation remains unchanged (no explicit κ_1, κ_2) for all CK spaces.

Taken altogether the constants $I_{\mathcal{P}_1^2}, I_{\mathcal{P}_2^2}, I_{\mathcal{P}_1\mathcal{P}_2}$ are the components of the general CK (symmetric) 'Fradkin tensor': in $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$

$$\begin{pmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{21} & \mathcal{F}_{22} \end{pmatrix} = \begin{pmatrix} I_{\mathcal{P}_1^2} & I_{\mathcal{P}_1\mathcal{P}_2} \\ I_{\mathcal{P}_1\mathcal{P}_2} & I_{\mathcal{P}_2^2} \end{pmatrix} = \begin{pmatrix} \mathcal{P}_1^2 + \mathcal{W}_{11}(q_1, q_2) & \mathcal{P}_1\mathcal{P}_2 + \mathcal{W}_{12}(q_1, q_2) \\ \mathcal{P}_1\mathcal{P}_2 + \mathcal{W}_{12}(q_1, q_2) & \mathcal{P}_2^2 + \mathcal{W}_{22}(q_1, q_2) \end{pmatrix}. \tag{53}$$

In the standard Euclidean case $\mathbf{E}^2$ these constants of motion reduce as they should to the ordinary Fradkin tensor [6, 5] (in agreement with (2)):

$$I_{\mathcal{P}_1^2}\big|_{\mathbf{E}^2} = \mathcal{P}_1^2 + \omega_0^2 x^2, \qquad I_{\mathcal{P}_1\mathcal{P}_2}\big|_{\mathbf{E}^2} = \mathcal{P}_1\mathcal{P}_2 + \omega_0^2 xy, \qquad I_{\mathcal{P}_2^2}\big|_{\mathbf{E}^2} = \mathcal{P}_2^2 + \omega_0^2 y^2. \tag{54}$$

Thus an essential property of the Euclidean harmonic oscillator, to have a tensor constant of motion, survives for the 'curved' harmonic oscillator in any CK space $\mathbf{S}^2_{\kappa_1[\kappa_2]}$. And the square of the angular momentum $\mathcal{J}^2 = I_{\mathcal{J}^2}$ is related to the determinant of the Fradkin tensor in a 'universal' way, explicitly independent of κ_1 and κ_2:

$$\det(\mathcal{F}) = \omega_0^2 I_{\mathcal{J}^2}. \tag{55}$$

Let us now see how to obtain the equation of the HO orbits in any CK space, directly from the superintegrable character. This only requires to enforce the universal fundamental relation (17) among the CK momenta taken from the conserved quantities $I_{\mathcal{J}^2}, I_{\mathcal{P}_1^2}, I_{\mathcal{P}_2^2}, I_{\mathcal{P}_1\mathcal{P}_2}$. Starting from (17) in the form $s^0\mathcal{J} = s^1\mathcal{P}_2 - s^2\mathcal{P}_1$, squaring it

$$(s^0)^2\mathcal{J}^2 = (s^1)^2\mathcal{P}_2^2 - 2s^1s^2\mathcal{P}_1\mathcal{P}_2 + (s^2)^2\mathcal{P}_1^2, \tag{56}$$

and drawing the three products $\mathcal{P}_1^2, \mathcal{P}_1\mathcal{P}_2, \mathcal{P}_2^2$ from the first integrals

$$\mathcal{P}_1^2 = \mathcal{F}_{11} - \mathcal{W}_{11}(q_1, q_2), \qquad \mathcal{P}_1\mathcal{P}_2 = \mathcal{F}_{12} - \mathcal{W}_{12}(q_1, q_2), \qquad \mathcal{P}_2^2 = \mathcal{F}_{22} - \mathcal{W}_{22}(q_1, q_2), \tag{57}$$

we get

$$(s^0)^2\mathcal{J}^2 = (s^1)^2(\mathcal{F}_{22} - \mathcal{W}_{22}) - 2s^1s^2(\mathcal{F}_{12} - \mathcal{W}_{12}) + (s^2)^2(\mathcal{F}_{11} - \mathcal{W}_{11}). \tag{58}$$

This relation does not involve any longer the CK momenta $\mathcal{P}_1, \mathcal{P}_2$ (which are *not* constant along the motion) but only the constant $\mathcal{J}, \mathcal{F}_{\mu\nu}$ and the coordinates; this is the orbit equation. But a further simplification is still possible, as it follows directly from (49):

$$(s^1)^2\mathcal{W}_{22} - 2s^1s^2\mathcal{W}_{12} + (s^2)^2\mathcal{W}_{11} = 0, \tag{59}$$

so the equation of the orbit reduces in terms of the ambient space coordinates (s^0, s^1, s^2) to

$$(s^0)^2\mathcal{J}^2 = (s^1)^2\mathcal{F}_{22} - 2s^1s^2\mathcal{F}_{12} + (s^2)^2\mathcal{F}_{11}, \tag{60}$$

which turns out to be, for all values of κ_1, κ_2, a CK conic in the intrinsic geometry of $\mathbf{S}^2_{\kappa_1[\kappa_2]}$, centred at the origin of the potential. By replacing the ambient coordinates by any other choice among (r, ϕ), (x, v) or (u, y) after (9), we get the orbit equation in that coordinate system. In particular, with polar coordinates, we get

$$\frac{1}{T^2_{\kappa_1}(r)} = \frac{\omega_0^2}{\mathcal{F}_{11}\mathcal{F}_{22} - \mathcal{F}_{12}^2}\left(\mathcal{F}_{11}C^2_{\kappa_2}(\phi) - 2\mathcal{F}_{12}C_{\kappa_2}(\phi)S_{\kappa_2}(\phi) + \mathcal{F}_{22}S^2_{\kappa_2}(\phi)\right). \tag{61}$$

As a consequence of the additional relation $\omega_0^2\mathcal{J}^2 = \mathcal{F}_{11}\mathcal{F}_{22} - \mathcal{F}_{12}^2$, the three quantities $\mathcal{F}_{11}, \mathcal{F}_{12}$ and $\mathcal{F}_{22}$ are a possible choice for a set of independent constants determining a particular orbit.

In all CK spaces, equation (61) includes ellipses with main semiaxis a, minor semiaxis b and orientation ϕ_0 of the main semiaxis relative to the chosen polar coordinate axis; its equation is

$$\frac{1}{T^2_{\kappa_1}(r)} = \frac{C^2_{\kappa_2}(\phi - \phi_0)}{T^2_{\kappa_1}(a)} + \frac{S^2_{\kappa_2}(\phi - \phi_0)}{T^2_{\kappa_1\kappa_2}(b)}, \tag{62}$$

whence by comparing with (60) we obtain the relation between the 'physical' constants $\mathcal{F}_{11}, \mathcal{F}_{12}$ and $\mathcal{F}_{22}$ and the geometric ones a, b and ϕ_0 for elliptic orbits. If we choose the direction of the main semiaxis as the origin of angles so that $\phi_0 = 0$ (this is always possible when $\kappa_2 > 0$; if $\kappa_2 < 0$ there is another generic possibility), the relevant relation is

$$\frac{1}{T^2_{\kappa_1}(r)} = \frac{\omega_0^2}{\mathcal{F}_{11}\mathcal{F}_{22}}\left(\mathcal{F}_{22}C^2_{\kappa_2}(\phi) + \mathcal{F}_{11}S^2_{\kappa_2}(\phi)\right) = \frac{C^2_{\kappa_2}(\phi)}{T^2_{\kappa_1}(a)} + \frac{S^2_{\kappa_2}(\phi)}{T^2_{\kappa_1\kappa_2}(b)}, \tag{63}$$

with

$$\mathcal{F}_{11} = \omega_0^2 T_{\kappa_1}^2(a) \equiv 2E_1, \qquad \mathcal{F}_{12} = 0, \qquad \mathcal{F}_{22} = \omega_0^2 T_{\kappa_1\kappa_2}^2(b) \equiv 2E_2, \tag{64}$$

and

$$\mathcal{J} = \omega_0 T_{\kappa_1}(a) T_{\kappa_1\kappa_2}(b), \qquad E = E_1 + \kappa_2 E_2 + \tfrac{1}{2}\kappa_1\kappa_2\mathcal{J}^2, \tag{65}$$

so that the eigenvalues of the Fradkin tensor are the two 'partial energies' of the two 1D harmonic oscillators in which the 2D system can be 'decomposed'.

Equation (61) may include orbits which are not CK 'ellipses', but other type of centred conics in the CK space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$; this depends on the space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ itself and eventually also on the range of the values of $\mathcal{F}_{\mu\nu}$. For instance, consider the hyperbolic plane $\mathbf{H}^2$ ($\kappa_1 < 0, \kappa_2 > 0$) where the two semiaxes appear in (62) through a hyperbolic tangent, which stays bounded for any real value of a; it is clear that the conics with $\mathcal{F}_{11} > \omega_0^2/-\kappa_1$ or $\mathcal{F}_{22} > \omega_0^2/-\kappa_1\kappa_2$ are not ellipses, as they cannot be cast in the form (62). In the hyperbolic plane, the family (61) of conics includes ellipses, equidistants and ultraellipses, depending on the values of the 'physical constants' $\mathcal{F}_{11}, \mathcal{F}_{12}, \mathcal{F}_{22}$, and even there exist regions in the $(\mathcal{F}_{11}, \mathcal{F}_{12}, \mathcal{F}_{22})$-space with no associated orbit at all.

By proceeding similarly with parallel coordinates, the equations of the oscillator orbits can be obtained in these coordinates. Of course, the relations between the 'physical constants' of the orbit $\mathcal{J}, \mathcal{F}_{11}, \mathcal{F}_{12}, \mathcal{F}_{22}$ and the geometric invariants of the conic remain unchanged, because they do not depend on the coordinates.

4.1. Momentum Hodographs for the harmonic oscillator

The complete symmetry between coordinates and momenta in the usual harmonic oscillator implies the well-known fact that Euclidean hodographs are also ellipses. This property also holds for the 'curved' harmonic oscillator in any CK space, as we show in this section. The derivation is quite simple. We start from the orbit equation (60) in the form

$$\mathcal{J}^2 = \mathcal{F}_{22}\left(\frac{s^1}{s^0}\right)^2 - 2\mathcal{F}_{12}\left(\frac{s^1}{s^0}\right)\left(\frac{s^2}{s^0}\right) + \mathcal{F}_{11}\left(\frac{s^2}{s^0}\right)^2. \tag{66}$$

Now we use (50) and rewrite the orbit equation as

$$\omega_0^2\mathcal{J}^2 = \mathcal{F}_{22}\mathcal{W}_{11} - 2\mathcal{F}_{12}\mathcal{W}_{12} + \mathcal{F}_{11}\mathcal{W}_{22}, \tag{67}$$

but the coordinate functions $\mathcal{W}_{ij}$ may be expressed in terms of the constant values of the Fradkin tensor and of the CK momenta themselves $\mathcal{W}_{ij} = \mathcal{F}_{ij} - \mathcal{P}_i\mathcal{P}_j$, hence by replacing we get

$$\omega_0^2\mathcal{J}^2 = \mathcal{F}_{22}\big(\mathcal{F}_{11} - \mathcal{P}_1^2\big) - 2\mathcal{F}_{12}\big(\mathcal{F}_{12} - \mathcal{P}_1\mathcal{P}_2\big) + \mathcal{F}_{11}\big(\mathcal{F}_{22} - \mathcal{P}_2^2\big), \tag{68}$$

where the terms quadratic in $\mathcal{F}_{ij}$ on the rhs add altogether to $2 \det F = 2\omega_0^2\mathcal{J}^2$. Simplifying we get the hodograph equation as

$$\omega_0^2\mathcal{J}^2 = \mathcal{F}_{22}\mathcal{P}_1^2 - 2\mathcal{F}_{12}\mathcal{P}_1\mathcal{P}_2 + \mathcal{F}_{11}\mathcal{P}_2^2. \tag{69}$$

In this equation $\mathcal{P}_1$ and $\mathcal{P}_2$ are the only variables; the remaining quantities are constants of motion. This represents an ellipse with centre at the origin in the plane $(\mathcal{P}_1, \mathcal{P}_2)$, whose natural geometry only keeps the track of the signature type κ_2 and is always flat, regardless of the value of the curvature κ_1 of the original CK space. Here, the ellipse semiaxes can be read directly as in the Euclidean plane. In the standard position where $\mathcal{F}_{12} = 0$ these semiaxes are

$$\frac{\omega_0\mathcal{J}}{\sqrt{\mathcal{F}_{22}}} = \frac{\omega_0\mathcal{J}}{\sqrt{2E_2}}, \qquad \frac{\omega_0\mathcal{J}}{\sqrt{\mathcal{F}_{11}}} = \frac{\omega_0\mathcal{J}}{\sqrt{2E_1}}. \tag{70}$$

The consistency of this result with all the previous ones is easily established.

5. Transforming the harmonic oscillator into the Kepler problem in curved spaces: the 'curved' Bohlin transform

The Euclidean configuration plane can be understood as a complex plane, $z = r\mathrm{e}^{\mathrm{i}\phi}$, in the standard way. Then the complex transformation $z \mapsto z^2$ maps origin-centred 'Hooke' ellipses (the orbits of an *attractive* harmonic oscillator) to focus-centred ellipses (the *negative energy* Keplerian orbits). Any Kepler ellipse can be obtained in this way, starting from a suitably chosen Hooke ellipse. Under the same map, Kepler hyperbolas can be obtained as images of the orbits of a *repulsive* harmonic oscillator, and Kepler parabolic orbits, which are limiting cases of both elliptic and hyperbolic orbits, appear as the images of straight lines, which are in turn the orbits of free motion, seen here as a 'zero strength' harmonic oscillator and hence a limiting case of both attractive and repulsive oscillators. Behind this fact, referred to (at least for Kepler ellipses) as the Bohlin theorem [38, 16] there is an interesting and intriguing duality among the linear and inverse square central force laws, already observed by Newton who found striking similarities between these two 'principal cases'. In its modern formulation (due originally to Kasner [39] and studied by Arnol'd [15]), this duality trades energy with the 'coupling constant' for the force (or the potential) and also extends to infinitely many other pairs of dual force laws with non-integer exponents, with r^{-4}, r^{-5}, r^{-7} as the only integer power force laws among dual pairs further to r and r^{-2} (see a very nice description in [16]).

An extension of the Bohlin transform to the constant curvature Riemannian spheres and pseudospheres exists [40, 41]. Hence a very natural question would be to ascertain whether there is a kind of 'general' Bohlin transform, relating the orbits of the *curved* harmonic oscillator and Kepler potentials within the full CK scheme, and also covering the parameter κ_2 and hence the constant curvature Lorentzian configuration spaces. We restrict here to a quite simple derivation of this 'general' Bohlin transform, without entering into the details, which would be worth of further discussion, specially in the Lorentzian case. Recall the polar form of the orbits for the Kepler potential (36), with the choice $\phi_0 = 0$:

$$T_{\kappa_1}(r) = \frac{\sqrt{\kappa_2}T_{\kappa_1\kappa_2}(p)}{1 + eC_{\kappa_2}(\phi)}, \tag{71}$$

and reexpress the polar equation for standard orbits of the harmonic oscillator (63), also with the choice $\phi_0 = 0$, in terms of the double angle, to obtain:

$$T^2_{\kappa_1}(r) = \frac{1}{D - GC_{\kappa_2}(2\phi)}, \qquad D = \frac{T^2_{\kappa_1}(a) + \kappa_2 T^2_{\kappa_1\kappa_2}(b)}{2\kappa_2 T^2_{\kappa_1}(a)T^2_{\kappa_1\kappa_2}(b)}, \qquad G = \frac{T^2_{\kappa_1}(a) - \kappa_2 T^2_{\kappa_1\kappa_2}(b)}{2\kappa_2 T^2_{\kappa_1}(a)T^2_{\kappa_1\kappa_2}(b)}, \tag{72}$$

(the notation conforms the one used in [20]). Then a simple comparison directly reveals the required 'curved' extension of the Bohlin transformation, using the same procedure working in the flat Euclidean case.

To describe this, denote by $\mathbb{C}_\eta$ the 'complex' plane defined as the set of numbers of the form $z = x + \mathrm{i}y$ with x, y real, $\mathrm{i}^2 = -\eta$, and with the natural extension for sums and products dictated by these rules, exactly as one does for the standard complex numbers. This endows $\mathbb{C}_\eta$ with a structure of commutative algebra which coincides with the ordinary complex algebra when $\eta = 1$ (or, after a simple rescaling, when $\eta > 0$). As a set $\mathbb{C}_\eta$ can be identified to $\mathbb{R}^2$ for any η, and the notation is chosen so that the standard algebra of complex numbers $\mathbb{C}$ is the member with $\eta = 1$ of the family $\mathbb{C}_\eta$, in analogy with the ordinary standard sphere $\mathbf{S}^2$ being the member with $\kappa_1 = 1, \kappa_2 = 1$ of the CK spaces $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$. The family $\mathbb{C}_\eta$ also encompasses two other algebraic systems: the double (or perplex or split-complex) numbers when $\eta < 0$, with the standard double numbers $\mathbb{C}_{-1} \equiv \mathbb{C}_-$ occurring precisely for $\eta = -1$, and the Study (or 'dual') numbers $\mathbb{C}_0$ when $\eta = 0$. The algebra $\mathbb{C}_\eta$ is a *composition algebra* for any η, but only when $\eta > 0$ it is a division algebra. The family of algebras $\mathbb{C}_\eta$ also appears naturally in relation to CK geometries, both real [42] as well as 'complex Hermitian' type ones [43].

Now consider the CK space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ and associate with the point with polar coordinates (r, ϕ) the 'complex' number $z := T_{\kappa_1}(r)e^{\mathrm{i}\phi} \in \mathbb{C}_{\kappa_2}$ (note that the type of these 'complex numbers' depends *only* on the signature type of the space, but not on its curvature). Under $z \mapsto z^2$, the modulus of z transform as $T_{\kappa_1}(r) \to T^2_{\kappa_1}(r)$, and the argument as $\phi \to 2\phi$, hence the mapping $z \mapsto z^2$ of $\mathbb{C}_\eta$ into itself carries curved Kepler orbits into curved harmonic oscillator ones, as evident from comparing (71) and (72). This is the 'curved' extension of the Bohlin transform. Note that both κ_1, κ_2 enter into the representation of points in the space $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ as 'complex' numbers in $\mathbb{C}_{\kappa_2}$; the curvature κ_1 appears explicitly in $T_{\kappa_1}(r)$ and the signature type κ_2 is implicit as minus the square of the imaginary unit i. The representation of points in Minkowskian $1 + 1$ spacetime as double numbers keeps the basic properties of the representation of points of the Euclidean plane as complex numbers and has been rediscovered many times. The existence of zero divisors in the algebra of double numbers is associated with the existence of points with vanishing separation to the origin in the Minkowskian plane.

It seems that this connection among the curved Kepler and oscillator, as well as the possibility of a more general 'curved' Kasner–Arnol'd duality are open questions worth of further study. A specific point would be to relate the Bohlin transform to the general 'curved' Levi-Civita regularization for the Kepler problem developed in full detail in [44].

6. Conclusions

Starting from the superintegrability of the 'curved' Kepler and the harmonic oscillator potential in any $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$ with constant curvature κ_1 and metric of signature type κ_2, either Riemannian, degenerate or Lorentzian, we have derived by purely algebraic means the equations of the *orbits* and the *momentum hodographs*. This derivation hinges on two sets of identities, holding similarly in all the CK spaces $\boldsymbol{S}^2_{\kappa_1[\kappa_2]}$. The identity in the first set relates the three Noether momenta in a 'universal way', explicitly independent of the potential and the two parameters κ_1, κ_2. The second set of identities relies on the superintegrability, and hence is specific to the particular superintegrable system; these identities relate some functions defined on the configuration space that appear in the quadratic constants of motion of I type. The interplay between these elements leads in a surprisingly direct way to the equations of both the orbits and the momentum hodographs for these systems. In particular, for the Kepler potential, the configuration space orbits are always CK conics with a focus at the potential origin, and the momentum hodographs are always CK cycles (or arcs of cycles) in the natural geometry of the $(\mathcal{P}_1, \mathcal{P}_2)$-plane; this provides a generalization of the celebrated result by Hamilton on the hodographs of the Kepler problem, a result which holds not only in the Euclidean configuration space, but *generically* within the family of CK spaces. In the harmonic oscillator, the interplay also leads very directly to the orbits, which are in all CK spaces conics centred at the origin, and to the momentum hodographs, which are always CK ellipses (or arcs of ellipses) in the natural geometry of the $(\mathcal{P}_1, \mathcal{P}_2)$-plane; this provides a generalization of the elementary well-known ellipse character of the Euclidean harmonic oscillator hodographs.

We have also mentioned, but not studied in detail, the 'curved' extension of the Bohlin transformation. The existence of striking similarities in the two Euclidean 'linear' and 'inverse square' central potentials was already observed by Newton, who referred to these as the two 'principal cases' [16]; indeed, it seems that all these striking similarities continue to hold for the curved harmonic oscillator and the Kepler problem in any CK configuration space.

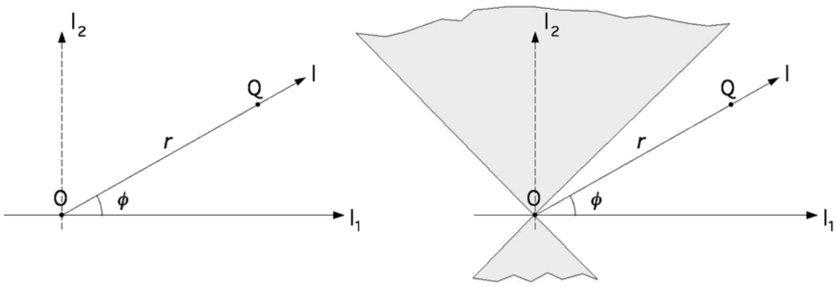

Figure A1. The 'polar' coordinates (r, ϕ). The diagram depicts the geometrical meaning of the polar coordinates (r, ϕ) in a general CK space $S^2_{\kappa_1[\kappa_2]}$, both in the locally Riemannian case $\kappa_2 > 0$ (left) and in the pseudo-Riemannian case $\kappa_2 < 0$ (right). In all the cases l_1, l_2, l are geodesics, and l_1, l_2 are orthogonal. The light cone through O is also shown in the Lorentzian diagram. The coordinate r has label κ_1 while ϕ has label κ_2. In the Riemannian case $\kappa_2 > 0$, the coordinate r is non-negative and vanishes at point O, where polar coordinates are singular and the angular coordinate ϕ ranges in the interval $[0, 2\pi/\sqrt{\kappa_2}]$ with periodic conditions. In the pseudo-Riemannian case r vanishes along the isotropes through O and would be pure imaginary in the shaded area with spacelike separation to O. In the unshaded area the angle ϕ (as depicted) ranges in the interval $[-\infty, \infty]$ while in the shaded area ϕ is of the form $\pi/(2\sqrt{\kappa_2}) + \tilde{\phi}$ with $\tilde{\phi} \in [-\infty, \infty]$. In both cases, for a given ϕ or $\tilde{\phi}$ the natural range of r involves positive as well as negative values. Note the natural interpretation of r and ϕ as canonical parameters of one-parameter subgroups of translations along the line l (conjugated to translations along l_1) or of rotations around the point O.

Acknowledgments

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Appendix A. The basic coordinates on two-dimensional Cayley–Klein manifolds

Consider the generic CK family of spaces $S^2_{\kappa_1[\kappa_2]}$, for any values of κ_1, κ_2. When κ_2 is positive a simple scaling may set $\kappa_1 = 1$, and within this choice the family includes the three constant curvature 2D Riemannian spaces $\mathbf{S}^2_{\kappa_1}, \mathbf{E}^2, \mathbf{H}^2_{\kappa_1}$. When κ_2 is negative, it may be reduced to -1 and within this choice the CK spaces are Lorentzian manifolds of constant curvature $\mathbf{AdS}^{1+1}_{\kappa_1}$, Minkowskian space $\mathbf{M}^{1+1}$ and de Sitter sphere $\mathbf{dS}^{1+1}_{\kappa_1}$. The coordinates employed in the paper, (r, ϕ), (u, y) or (x, v), can be defined in the following way (the construction may be easily visualized in the case of the sphere; for more details, refer to [20]): choose a point O a point on $S^2_{\kappa_1[\kappa_2]}$, to be considered as the origin point, and let l_1 be an oriented geodesic (timelike if κ_2 is non-positive, generated by P_1) through O and l_2 the oriented geodesic orthogonal to l_1 through O (hence spacelike if κ_2 is non-positive and generated by P_2). For a generic point Q (in some suitable domains), consider the geodesic l joining O with Q, the geodesic l_2' through Q and orthogonal to l_1, intersecting l_1 at Q_1, and the geodesic l_1' through Q and orthogonal to l_2, intersecting l_2 at Q_2. In terms of these geometric constructions the coordinates (r, ϕ), (u, y) or (x, v) are defined as follows.

The (geodesic) polar coordinates (r, ϕ) of Q relative to the origin O and the positive geodesic ray of l_1 are the distance r between Q and O measured along l, and the angle ϕ between l and the positive ray l_1 measured around O (Figure A1).

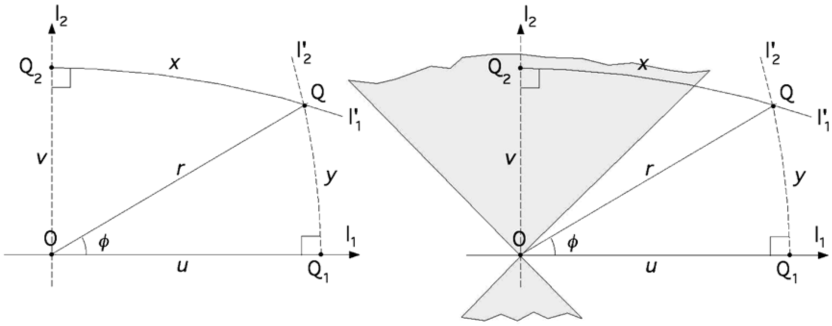

Figure A2. The 'parallel' coordinates (u, y) and (v, x). The diagram depicts the geometrical meaning of the coordinates (u, y) and (v, x), for the same situation and with the same conventions as in figure A1. The lines l'_1, l'_2 are geodesics through Q orthogonal to l_2, l_1, respectively. The coordinates u, x have label κ_1 and are defined near O in both the Riemannian and pseudo-Riemannian cases. The coordinates v, y have label $\kappa_1\kappa_2$ and the corresponding geodesics are represented as dashed; in the pseudo-Riemannian case this means these geodesics are spacelike. In all cases the ordinary sign conventions applies. Note the natural interpretation of all coordinates as canonical parameters of some elements in one-parameter subgroups of translations along the lines l_1, l_2, l'_1, l'_2 or of rotations around the point O, relatively to some generators whose scaling is already fixed; in the locally Lorentzian case the quantities v, y, whose label is $\kappa_1\kappa_2$, are related to the length as determined by the metric in the space $S^2_{\kappa_1[\kappa_2]}$ by a factor $\sqrt{\kappa_2}$.

The (geodesic) parallel coordinates (u, y) of Q relative to the axes l_1, l_2 are defined as follows. The coordinate u is the canonical parameter of the element in the one-dimensional subgroup of translations along l_1, generated by P_1 and with label κ_1, which brings O to Q_1; this value coincides with the distance between O and Q_1 computed along l_1 with the CK space metric (here as in the previous case, this canonical parameter is determined uniquely, as the generator P_1 is precisely determined by the commutation relations (4) and cannot by scaled without changing κ_1, κ_2 which are taken here as fixed from the outset). The coordinate y is the canonical parameter of the element in the one-dimensional subgroup of translations along l'_2, generated by $\mathrm{e}^{uP_1} P_2\, \mathrm{e}^{-uP_1}$ and with label $\kappa_1\kappa_2$, which brings Q_1 to Q; this value is related with the distance between Q_1 and Q computed along l'_1 with the CK space metric by a factor $\sqrt{\kappa_2}$ (note that in the Lorentzian case y is always real, yet both a spacelike separation and $\sqrt{\kappa_2}$ are pure imaginary), refer to figure A2.

The (geodesic) parallel coordinates (v, x) of Q relative to the axes l_1, l_2 are defined as follows. The coordinate v is the canonical parameter of the element in the one-dimensional subgroup of translations along l_2, generated by P_2 and with label $\kappa_1\kappa_2$, which brings O to Q_2; this value is related with the distance between O and Q_2 computed along l_2 with the CK space metric by a factor $\sqrt{\kappa_2}$, so that v is always real even in the Lorentzian case. The coordinate x is the canonical parameter of the element in the one-dimensional subgroup of translations along l'_1, generated by $\mathrm{e}^{vP_2} P_1\, \mathrm{e}^{-vP_2}$ and with label κ_1, which brings Q_2 to Q; this value coincides with the distance between Q_2 and Q computed along l'_1 with the CK space metric.

As a pertinent final remark, we stress that in the general $\kappa_1 \neq 0$ case, we have $u \neq x$ and $v \neq y$; only in flat spaces (whether Euclidean or Minkowskian) do the equalities $x = u$, $v = y$ hold.

Appendix B. Conics in spaces of constant curvature

This is only a reminder of the basics. For more details in the general CK case with arbitrary κ_1, κ_2, see [20]. The conics in the three Riemannian cases (with $\kappa_2 = 1$) are also discussed in [34].

The *geometric* definition of conics makes sense in any 2D space of constant curvature κ_1 and non-degenerate metric ($\kappa_2 \neq 0$) and involves *focal elements*, i.e., either points which are assumed oriented or lines (geodesics of the intrinsic CK metric) which we assume to be oriented and cooriented. There are *three* generic types of conics in a CK space $\mathbf{S}^2_{\kappa_1[\kappa_2]}$ defined as follows.

An *ellipse/hyperbola* will be the set of points with a constant sum/difference $2a$ of distances r_1, r_2 to two fixed points F_1, F_2 separated by a distance $2f$ and called *foci*.

A *parabola* will be the set of points with a constant sum/difference of distances $r_1, \tilde{r}_2$ to a fixed point F_1, called *focus*, and to a fixed line f_2, called *focal line* ($\tilde{r}_2$ is assumed to be oriented); the *oriented* distance between F_1 and f_2 plays here the role of focal separation.

An *ultraellipse/ultrahyperbola* will be the set of points with a constant sum/difference $2a$ of oriented distances $\tilde{r}_1, \tilde{r}_2$ to two fixed intersecting lines f_1, f_2 separated by an angle $2F$ and called *focal lines*.

These three pairs of curves, each pair sharing the same focal elements, are the *generic* conics in the generic CK 2D space of constant curvature $\kappa_1 \neq 0, \kappa_2 \neq 0$, and all the remaining possible conics are either particular instances with focal separation vanishing ($f = 0, \phi = 0, F = 0$) or limiting cases, where some conic elements go to infinity (if possible at all); both particular and limiting cases can be obtained as suitable limits from the generic conics.

In particular, ellipses with vanishing focal separation are circles with centre at the (double) focus, and ultraellipses with vanishing focal angle separation are equidistant curves with baseline the (double) focal line. Circles and equidistant curves are curves of constant geodesic curvature. On the sphere circles and equidistant curves coincide; this is clear on a sphere in geographic coordinates, where these curves correspond to the constant latitude lines, which are circles with centre at the pole and equidistants with base at the equator. In the hyperbolic plane, circles and equidistants are different curves, and they have a common limiting case, the horocycles. Collectively, these curves of constant geodesic curvature are called *cycles*.

The Kepler orbits are conics with a focus at the origin. Their equation involves two basic parameters, the eccentricity e and the semilatus rectum p. Eccentricity is related to the major semiaxis $2a$ and to the focal distance $2f$, both quantities with label κ_1, in the general CK case, as $e = S_{\kappa_1}(2f)/S_{\kappa_1}(2a)$, reducing to the Euclidean $e = f/a$ when $\kappa_1 = 0$. If the conic is placed on its standard position, with the points on the conic with stationary distance to the origin placed on the basic line l_1, then the semilatus rectum is the y coordinate of the point on the conic on the line l_2; note this quantity has label $\kappa_1\kappa_2$ and its value is pure imaginary when $\kappa_2 < 0$, so that the combination $\sqrt{\kappa_2}T_{\kappa_1\kappa_2}(y)$ appearing in the conic equation is always real.

Harmonic oscillator orbits are conics with centre at the origin. In this case, the best choice is to write the general equation in terms of two basic parameters, the two principal semiaxes, denoted here as a, with label κ_1, and b, with label $\kappa_1\kappa_2$.

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